Proof of all-order finiteness for planar β-deformed Yang-Mills

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ABSTRACT: We study a marginal deformation of $\mathcal{N} = 4$ Yang-Mills, with a real deformation parameter β. This β-deformed model has only $\mathcal{N} = 1$ supersymmetry and a U(1)×U(1) flavor symmetry. The introduction of a new superspace $\star$-product allows us to formulate the theory in $\mathcal{N} = 4$ light-cone superspace, despite the fact that it has only $\mathcal{N} = 1$ supersymmetry. We show that this deformed theory is conformally invariant, in the planar approximation, by proving that its Green functions are ultra-violet finite to all orders in perturbation theory.

Contents

1. Introduction

One of the most striking consequences of supersymmetry is an improvement in the ultra-violet behavior of quantum field theories. This feature provides one of the main motivations for the study of supersymmetric models in connection with the resolution of the hierarchy problem.

The maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills [1] theory is particularly remarkable in that it is ultra-violet finite [2 – 4], therefore representing an example of a four-dimensional quantum field theory which is conformally invariant at the quantum level. The discovery of the special properties of the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory has led to extensive investigations aimed at identifying field theories with the same finiteness properties, but a smaller amount of supersymmetry [5]. A special class of such theories are those obtained as exactly marginal deformations of $\mathcal{N} = 4$ SYM. These were classified in [6], where it was argued that there exists a two (complex) parameter family of such deformations. In this paper we shall provide a proof of finiteness, to all orders in planar perturbation theory, for a particular model in this class, which involves a single real deformation parameter.

The existence of families of marginal deformations of $\mathcal{N} = 4$ SYM has interesting consequences in the context of the AdS/CFT correspondence [7]. In the framework of the AdS/CFT duality the conformal symmetry of the boundary field theory is associated with the isometry group of the dual anti-de Sitter background. Therefore marginal deformations of $\mathcal{N} = 4$ SYM should correspond to deformations of the $AdS_5 \times S^5$ background which preserve the SO(4,2) group of isometries of the AdS_5 factor. This correspondence was first considered in [8] and further studied in [9], where, by expanding the supergravity equations around the $AdS_5 \times S^5$ solution, it was found that the dimension of the space of solutions preserving $\mathcal{N} = 1$ supersymmetry as well as the SO(4,2) isometry group is the same as that of the space of marginal deformations of $\mathcal{N} = 4$ SYM [6]. An exact supergravity solution dual to a $\mathcal{N} = 1$ superconformal field theory with a U(1)×U(1) flavor symmetry was constructed in [10]. The theory dual to the supergravity background of [10] belongs to the class of marginal deformations of $\mathcal{N} = 4$ SYM referred to as β-deformations, which are characterized, in $\mathcal{N} = 1$ superspace, by a superpotential of the form

$$W = \int d^4x \left[\int d^2\theta\, g\, h\, \text{Tr} \left(e^{i\pi\beta} \Phi^1 \Phi^2 \Phi^3 - e^{-i\pi\beta} \Phi^1 \Phi^3 \Phi^2 \right) + \text{h.c.} \right], \qquad (1.1)$$

where g is the standard Yang-Mills coupling and h and β are two complex deformation parameters. In (1.1) Φ^1, Φ^2 and Φ^3 are three $\mathcal{N} = 1$ chiral superfields. Following [10] the perturbative properties of theories in this class have been extensively studied [11 – 14].

In this paper we focus on the special case in which the superpotential (1.1) has $h = 1$ and $\beta \in \mathbb{R}$. We prove that this deformation of $\mathcal{N} = 4$ SYM is conformally invariant in the planar limit by showing that all the Green functions in the theory are finite. In order to prove this result to all orders in perturbation theory we shall realize the deformation by the introduction of new star-products acting in superspace. This will allow us to formulate the theory in $\mathcal{N} = 4$ light-cone superspace and to use the same arguments utilized in [3, 4] to prove the ultra-violet finiteness of $\mathcal{N} = 4$ SYM. For an introduction to light-cone superspace, we refer the reader to [15 – 17]. The super-Poincaré and superconformal algebras were presented in light-cone superspace in references [16 – 19]. The truncation of supersymmetry and superfields in this context was discussed in [20, 21].

The formulation of the deformed Yang-Mills theory using star-products presents analogies with the construction of non-commutative field theories [22]. However, the theory that

Field	$U(1)_1$	$U(1)_2$
ϕ^1, λ^1	0	-1
ϕ^2, λ^2	$+1$	$+1$
ϕ^3, λ^3	-1	0

Table 1: Flavor charges of the fields in β-deformed $\mathcal{N} = 4$ Yang-Mills.

we study is an ordinary gauge theory and these analogies are purely formal. In particular, we stress that the β-deformed Yang-Mills theory retains a non-trivial dependence on the deformation parameter in the planar limit.

As will be explicitly shown, the proof of [3, 4] can only be applied to the deformed model in the planar approximation. The analysis that we present is valid for arbitrary gauge group, but the case of $\mathrm{SU}(N)$ is particularly interesting, since in this case the theory is expected to be dual to the background of [10]. The all-order finiteness of the β-deformed Yang-Mills theory in the planar limit suggests that the inclusion of string tree-level corrections should not break the SO(4,2) isometries of the supergravity solution of [10]. Our formalism does not allow us to reach any conclusion about the finiteness of the theory for finite N and thus about the effect of string loop corrections on the dual background.

This paper is organized as follows. In sections 2 and 3 we show how to formulate the β-deformed theory in $\mathcal{N} = 4$ light-cone superspace. Section 4 presents the proof of finiteness to all orders in perturbation theory following the $\mathcal{N} = 4$ analysis of [3, 4]. Various appendices discuss aspects of the superspace calculations which are affected by the introduction of star-products.

2. β-deformed Yang-Mills

The β-deformed Yang-Mills theory has the same field content as $\mathcal{N} = 4$ Yang-Mills. In terms of $\mathcal{N} = 1$ multiplets, it consists of three chiral multiplets, $(\phi^I, \lambda_\alpha^I)$, $I = 1, 2, 3$ in the adjoint representation of the gauge group and a vector multiplet, $(A_\mu, \lambda_\alpha^4)$. The deformed theory has only $\mathcal{N} = 1$ supersymmetry and, in addition to the U(1) R-symmetry, it has a U(1)$\times$U(1) flavor symmetry. The three complex scalars, ϕ^I, and the three Weyl fermions, λ_α^I, are charged under the U(1)$\times$U(1) symmetry while the fields in the vector multiplet are not. The deformation is realized simply by replacing all the commutators of charged fields in the $\mathcal{N} = 4$ Yang-Mills action by $*$-commutators [10]. The $*$-commutator is defined by

$$[f, g]_* = f * g - g * f \,, \tag{2.1}$$

where

$$f * g = e^{i\pi\beta(Q_f^1 Q_g^2 - Q_f^2 Q_g^1)} fg \,. \tag{2.2}$$

The charges, Q^1 and Q^2, of the various fields with respect to the two U(1) flavor symmetries are read off from table 1.

The β-deformation breaks the $\mathcal{N}=4$ supersymmetry down to $\mathcal{N}=1$. The resulting deformed action reads

$$
\mathcal{S} = \int \mathrm{d}^4 x \, \mathrm{Tr} \left\{ \frac{1}{2} F^{\mu\nu} F_{\mu\nu} + 2(D^\mu \bar{\phi}_I)(D_\mu \phi^I) - 2i\, \lambda^I \not{D} \bar{\lambda}_I - 2i\, \lambda^4 \not{D} \bar{\lambda}_4 \right.
$$
$$
- 2\sqrt{2}\, g \left[\left(\epsilon_{IJK}[\lambda^I, \lambda^J]_* \phi^K + \epsilon^{IJK}[\bar{\lambda}_I, \bar{\lambda}_J]_* \bar{\phi}_K \right) - \left([\lambda^4, \lambda^I]\bar{\phi}_I + [\bar{\lambda}_4, \bar{\lambda}_I]\phi^I \right) \right]
$$
$$
\left. + g^2 \left(\frac{1}{2} [\phi^I, \bar{\phi}_I][\phi^J, \bar{\phi}_J] - [\phi^I, \phi^J]_* [\bar{\phi}_I, \bar{\phi}_J]_* \right) \right\} . \tag{2.3}
$$

In this paper, we restrict ourselves to the case where the deformation parameter β is real. This ensures that the action as written in (2.3) is real.

In the following we will formulate the theory in $\mathcal{N}=4$ light-cone superspace and so it is convenient to use a manifestly SU(4) notation. We introduce scalar fields, φ^{mn}, $m, n = 1, 2, 3, 4$, satisfying

$$
\varphi^{mn} = -\varphi^{nm}, \qquad \bar{\varphi}_{mn} = (\varphi^{mn})^\dagger = \frac{1}{2} \epsilon_{mnpq} \varphi^{pq} \tag{2.4}
$$

and related to the ϕ^I's by

$$
\phi^I = 2\, \varphi^{I4}, \qquad \bar{\phi}_I = \epsilon_{IJK4}\, \varphi^{JK}, \qquad I, J, K = 1, 2, 3. \tag{2.5}
$$

The fermion fields are combined into

$$
\lambda_\alpha^m = (\lambda_\alpha^I, \lambda_\alpha^4). \tag{2.6}
$$

3. The light-cone formalism

We now proceed to formulate the β-deformed theory of the previous section in the light-cone gauge. With the space-time metric $(-, +, +, +)$, the light-cone coordinates and their derivatives are

$$
\begin{aligned}
x^\pm &= \tfrac{1}{\sqrt{2}}(x^0 \pm x^3), & x &= \tfrac{1}{\sqrt{2}}(x^1 + ix^2), & \bar{x} &= \tfrac{1}{\sqrt{2}}(x^1 - ix^2), \\
\partial_\pm &= \tfrac{1}{\sqrt{2}}\left(\tfrac{\partial}{\partial x^0} \pm \tfrac{\partial}{\partial x^3} \right), & \bar{\partial} &= \tfrac{1}{\sqrt{2}}\left(\tfrac{\partial}{\partial x^1} - i\tfrac{\partial}{\partial x^2} \right), & \partial &= \tfrac{1}{\sqrt{2}}\left(\tfrac{\partial}{\partial x^1} + i\tfrac{\partial}{\partial x^2} \right).
\end{aligned} \tag{3.1}
$$

3.1 The light-cone component description

The choice of light-cone gauge involves eliminating the unphysical degrees of freedom. In the case of the gauge field, which splits into light-cone components, A_+, A_-, A and $\bar{A}$ this involves setting

$$
A_- = 0 \tag{3.2}
$$

and using the equations of motion to solve for A_+. On the light-cone, the fermions split up as

$$
\lambda_\alpha^m \rightarrow (\chi^{m(+)}, \chi^{m(-)}). \tag{3.3}
$$

The equations of motion allow us to eliminate $\chi^{m(+)}$. For simplicity of notation, we drop the $(-)$ index from the physical component, $\chi^{m(-)}$. For details regarding the derivation of the light-cone action in the non-deformed case, we refer the reader to [15].

The introduction of $*$-commutators into the action does not affect the validity of the light-cone procedure. We stress that this point is non-trivial because of the non-linearity of the equations of motion. In appendix A, we list the properties of the $*$-product that ensure that this procedure remains unaffected. We find that the light-cone component action describing β-deformed Yang-Mills is

$$
\begin{aligned}
S = \int \mathrm{d}^4 x \, \mathrm{Tr} \Bigg\{ & 2\bar{A}\Box A + \frac{1}{2}\bar{\varphi}_{mn}\Box\varphi^{mn} - \frac{2i}{\sqrt{2}}\bar{\chi}_m\frac{\Box}{\partial_-}\chi^m \\
& + g\left[2i\frac{\bar{\partial}}{\partial_-}A[\partial_-\bar{A}, A]_* + \frac{i}{2}\frac{\bar{\partial}}{\partial_-}A[\partial_-\bar{\varphi}_{mn}, \varphi^{mn}]_* - \frac{i}{2}A[\bar{\partial}\bar{\varphi}_{mn}, \varphi^{mn}]_* \right. \\
& \quad\left. -\sqrt{2}\frac{\bar{\partial}}{\partial_-}A[\bar{\chi}_m, \chi^m]_* + \sqrt{2}\,A\left[\chi^m, \frac{\bar{\partial}}{\partial_-}\bar{\chi}_m\right]_* - \sqrt{2}\frac{\bar{\partial}}{\partial_-}\bar{\chi}_m[\bar{\chi}_n, \varphi^{mn}]_* + \mathrm{h.c.}\right] \\
& + g^2\left[4\frac{1}{\partial_-}[\partial_- A, \bar{A}]_*\frac{1}{\partial_-}[\partial_-\bar{A}, A]_* + [\varphi^{mn}, A]_*[\bar{\varphi}_{mn}, \bar{A}]_* \right. \\
& \quad + \frac{1}{\partial_-}[\partial_-\bar{A}, A]_*\frac{1}{\partial_-}[\partial_-\bar{\varphi}_{mn}, \varphi^{mn}]_* + \frac{1}{\partial_-}[\partial_- A, \bar{A}]_*\frac{1}{\partial_-}[\partial_-\bar{\varphi}_{mn}, \varphi^{mn}]_* \\
& \quad + \frac{1}{8}[\varphi^{mn}, \varphi^{pq}]_*[\bar{\varphi}_{mn}, \bar{\varphi}_{pq}]_* + \frac{1}{4}\frac{1}{\partial_-}[\partial_-\bar{\varphi}_{mn}, \varphi^{mn}]_*\frac{1}{\partial_-}[\partial_-\bar{\varphi}_{pq}, \varphi^{pq}]_* \\
& \quad - i2\sqrt{2}\frac{1}{\partial_-}[\bar{\chi}_m, \bar{A}]_*[A, \chi^m]_* + i2\sqrt{2}\frac{1}{\partial_-}[\chi^m, A]_*[\bar{\varphi}_{mn}, \chi^n]_* \\
& \quad + i2\sqrt{2}\frac{1}{\partial_-}[\bar{\chi}_m, \bar{A}]_*[\varphi^{mn}, \bar{\chi}_n]_* + i2\sqrt{2}\frac{1}{\partial_-}[\bar{\chi}_m, \varphi^{mn}]_*[\bar{\varphi}_{np}, \chi^p]_* \\
& \quad + i2\sqrt{2}\frac{1}{\partial_-}[\partial_- A, \bar{A}]_*\frac{1}{\partial_-}[\bar{\chi}_m, \chi^m]_* + i2\sqrt{2}\frac{1}{\partial_-}[\partial_-\bar{A}, A]_*\frac{1}{\partial_-}[\bar{\chi}_m, \chi^m]_* \\
& \quad\left.\left. + i\sqrt{2}\frac{1}{\partial_-}[\partial_-\bar{\varphi}_{mn}, \varphi^{mn}]_*\frac{1}{\partial_-}[\bar{\chi}_p, \chi^p]_* - 2\frac{1}{\partial_-}[\bar{\chi}_m, \chi^m]_*\frac{1}{\partial_-}[\bar{\chi}_n, \chi^n]_* \right]\right\}, \quad (3.4)
\end{aligned}
$$

where we use for the $\frac{1}{\partial_-}$ operator the prescription given in [3]. As in the covariant case, this light-cone component action is obtained by replacing all the commutators in the $\mathcal{N}=4$ light-cone action [15] by $*$-commutators. Notice, however, that many of the $*$-commutators are actually ordinary commutators because of charge neutrality.

3.2 Light-cone superspace formalism

The β-deformed theory has $\mathcal{N}=1$ supersymmetry. Despite this we will show that the theory can be formulated in $\mathcal{N}=4$ light-cone superspace thanks to the fact that its field content is identical to that of $\mathcal{N}=4$ Yang-Mills. This is achieved by introducing a new star product in superspace which implements the effects of the deformation introduced into the component action. The $\mathcal{N}=4$ light-cone superspace $[15-21]$ is made up of four bosonic coordinates, $x^+, x^-, x, \bar{x}$, and eight fermionic coordinates,[1] $\theta^m, \bar{\theta}_m$, $m=1,2,3,4$. These will be collectively denoted by $z=(x^+, x^-, x, \bar{x}, \theta^m, \bar{\theta}_m)$.

All the degrees of freedom of the deformed theory are described by a single scalar superfield [15]. This superfield is defined by the constraints

$$
d^m \Phi = 0, \qquad \bar{d}_n \bar{\Phi} = 0, \tag{3.5}
$$

---

[1]These Grassmann coordinates do not carry spinor indices.

Variable	$U(1)_1$	$U(1)_2$
θ^1	0	-1
θ^2	$+1$	$+1$
θ^3	-1	0
θ^4	0	0

Table 2: θ charges under the flavor symmetry.

as well as the "inside-out constraints"

$$\bar{d}_m \bar{d}_n \Phi = \frac{1}{2} \epsilon_{mnpq} d^p d^q \bar{\Phi}, \tag{3.6}$$

where $\bar{\Phi}$ is the complex conjugate of Φ. The superspace chiral derivatives in the above expressions are

$$d^m = -\frac{\partial}{\partial \bar{\theta}_m} + \frac{i}{\sqrt{2}} \theta^m \partial_-, \qquad \bar{d}_n = \frac{\partial}{\partial \theta^n} - \frac{i}{\sqrt{2}} \bar{\theta}_n \partial_- \tag{3.7}$$

and obey

$$\{d^m, \bar{d}_n\} = i\sqrt{2}\, \delta^m_n \partial_- \,. \tag{3.8}$$

The superfield satisfying the constraints (3.5) and (3.6) is [15]

$$\Phi(x, \theta, \bar{\theta}) = -\frac{1}{\partial_-} A(y) - \frac{i}{\partial_-} \theta^m \bar{\chi}_m(y) + \frac{i}{\sqrt{2}} \theta^m \theta^n \overline{\varphi}_{mn}(y)$$
$$+ \frac{\sqrt{2}}{6} \theta^m \theta^n \theta^p \epsilon_{mnpq} \chi^q(y) - \frac{1}{12} \theta^m \theta^n \theta^p \theta^q \epsilon_{mnpq} \partial_- \bar{A}(y), \tag{3.9}$$

where $y = (x, \bar{x}, x^+, y^- \equiv x^- - \frac{i}{\sqrt{2}} \theta^m \bar{\theta}_m)$ is the chiral coordinate and the r.h.s. of (3.9) is understood as a power expansion around x^-.

3.2.1 The superspace $\star$-product

The deformation in the component action is realized using the $*$-commutator. In order to formulate the deformed theory in superspace, we introduce a new superspace $\star$-product whose effect on superfields mimics the action of the $*$-product on component fields.

Star-products differ from ordinary products by phase factors. In the component formulation, these phase factors were associated with the charges carried by the component fields [10]. In superspace, we will think of these phase factors as coming from the θ's instead. This is possible because each component field in a superfield is accompanied by a unique combination of θ's. Based on table 1 we assign to the θ variables (the charges of the $\bar{\theta}$'s are opposite to those of the θ's) the U(1)×U(1) charges in table 2.

With this assignment the superspace $\star$-product is realized in terms of operators which count the number of θ's and $\bar{\theta}$'s.

We define the $\star$-product of two superfields, F and G, by

$$F \star G = F\, e^{i\pi\beta(\overleftarrow{Q}^1_F \overrightarrow{Q}^2_G - \overleftarrow{Q}^2_F \overrightarrow{Q}^1_G)}\, G\,, \tag{3.10}$$

where the charges are the operators

$$
\begin{aligned}
\overrightarrow{Q}^1 &= \theta^2\frac{\overrightarrow{\partial}}{\partial\theta^2} - \theta^3\frac{\overrightarrow{\partial}}{\partial\theta^3} - \bar\theta_2\frac{\overrightarrow{\partial}}{\partial\bar\theta_2} + \bar\theta_3\frac{\overrightarrow{\partial}}{\partial\bar\theta_3}\,,\\
\overleftarrow{Q}^1 &= \frac{\overleftarrow{\partial}}{\partial\theta^2}\theta^2 - \frac{\overleftarrow{\partial}}{\partial\theta^3}\theta^3 - \frac{\overleftarrow{\partial}}{\partial\bar\theta_2}\bar\theta_2 + \frac{\overleftarrow{\partial}}{\partial\bar\theta_3}\bar\theta_3\,,\\
\overrightarrow{Q}^2 &= \theta^2\frac{\overrightarrow{\partial}}{\partial\theta^2} - \theta^1\frac{\overrightarrow{\partial}}{\partial\theta^1} - \bar\theta_2\frac{\overrightarrow{\partial}}{\partial\bar\theta_2} + \bar\theta_1\frac{\overrightarrow{\partial}}{\partial\bar\theta_1}\,,\\
\overleftarrow{Q}^2 &= \frac{\overleftarrow{\partial}}{\partial\theta^2}\theta^2 - \frac{\overleftarrow{\partial}}{\partial\theta^1}\theta^1 - \frac{\overleftarrow{\partial}}{\partial\bar\theta_2}\bar\theta_2 + \frac{\overleftarrow{\partial}}{\partial\bar\theta_1}\bar\theta_1\,.
\end{aligned}
\tag{3.11}
$$

The various terms in the θ-expansion of a superfield, as in (3.9), have definite U(1)×U(1) charges. Therefore, after substituting the θ-expansion in the $\star$-product of two superfields, the operators in (3.11) acting on each term in the sum produce definite phases according to the charge assignments in table 2. This is useful as it makes the manipulation of $\star$-products in superspace expressions much simpler. Notice that, although in the $\mathcal{N}=4$ light-cone superspace formulation the phase factors introduced by the $\star$-products are associated with the θ^m and $\bar\theta_m$ fermionic coordinates, the U(1)×U(1) symmetry is an ordinary flavor symmetry. This is to be distinguished from the $U(1)_R$ symmetry which is the standard $\mathcal{N}=1$ R-symmetry and acts both on the θ's and $\bar\theta$'s and on the component fields.

3.2.2 The action

The light-cone superspace action for β-deformed Yang-Mills is[2]

$$
\begin{aligned}
S = 72\int \mathrm{d}^4x \int \mathrm{d}^4\theta\,\mathrm{d}^4\bar\theta\,\mathrm{Tr}\Bigg\{ &-2\,\bar\Phi\,\frac{\Box}{\partial_-^2}\,\Phi + i\,\frac{8}{3}\,g\left(\frac{1}{\partial_-}\bar\Phi\,[\Phi,\bar\partial\Phi]_\star + \frac{1}{\partial_-}\Phi\,[\bar\Phi,\partial\bar\Phi]_\star\right)\\
&+ 2\,g^2\left(\frac{1}{\partial_-}[\Phi,\partial_-\Phi]_\star\,\frac{1}{\partial_-}[\bar\Phi,\partial_-\bar\Phi]_\star + [\Phi,\bar\Phi]_\star\,[\Phi,\bar\Phi]_\star\right)\Bigg\}\,.
\end{aligned}
\tag{3.12}
$$

Expanding the $\star$-commutators and performing the Grassmann integrations reproduces exactly (3.4). This justifies our definition of the superspace $\star$-product. This deformed theory is formulated in a manifestly $\mathcal{N}=4$ superspace, but because of the presence of the $\star$-products only one supersymmetry remains unbroken. We stress that this action can be obtained from the light-cone superspace action of [15] by just replacing ordinary commutators by $\star$-commutators.

The action in (3.12) can be expressed purely in terms of the chiral superfield. This is possible using the inside-out constraint in (3.6) which implies that

$$
\bar\Phi = \frac{1}{48}\frac{\bar d^4}{\partial_-^2}\Phi\,,
\tag{3.13}
$$

where

$$
\bar d^4 = \varepsilon^{mnpq}\,\bar d_m\bar d_n\bar d_p\bar d_q\,.
\tag{3.14}
$$

---

[2]This is an ordinary "commutative" field theory although the definition in (3.10) suggests the introduction of non-commutativity.

Unless otherwise indicated this notation will be used in all the following formulae. Similarly we will use

$$d^4 = \varepsilon_{mnpq} \, d^m d^n d^p d^q \,. \tag{3.15}$$

The rules for partially integrating chiral derivatives in the action are modified due to the presence of the $\star$-products. These modified manipulations of the chiral derivatives are explained in appendix A.2.

From the kinetic term in the action, we read off the propagator (in this equation, we make explicit the matrix indices on Φ)

$$\langle (\Phi)^u{}_v(z_1) \, (\Phi)^r{}_s(z_2) \rangle = \langle \Phi^a(z_1) \, (T^a)^u{}_v \, \Phi^b(z_2) \, (T^b)^r{}_s \rangle = \Delta^u{}_v{}^r{}_s (z_1 - z_2) \,, \tag{3.16}$$

where $z = (x^+, x^-, x, \bar{x}, \theta, \bar{\theta})$ and T^a, T^b are representation matrices for the Lie algebra. The corresponding momentum-space propagator reads

$$\Delta^u{}_v{}^r{}_s (k, \theta_{(1)}, \bar{\theta}_{(1)}, \theta_{(2)}, \bar{\theta}_{(2)}) = t^u{}_v{}^r{}_s \frac{1}{k_\mu^2} \, d^4_{(1)} \, \delta^8(\theta_{(1)} - \theta_{(2)}) \,, \tag{3.17}$$

where $\theta_{(1)}$ and $\theta_{(2)}$ denote the fermionic coordinates at points z_1 and z_2 respectively and $t^u{}_v{}^r{}_s$ is a tensor whose precise structure depends on the choice of gauge group. In our calculations, we use matrix notation and this tensor will not appear explicitly. The fermionic δ-function is

$$\delta^8(\theta_{(1)} - \theta_{(2)}) = (\theta_{(1)} - \theta_{(2)})^4 \, (\bar{\theta}_{(1)} - \bar{\theta}_{(2)})^4 \,. \tag{3.18}$$

In appendix B, we further streamline our notation and display a sample Wick contraction.

4. Proof of finiteness

In this section we explicitly prove that all light-cone superspace Green functions in the β-deformed theory are finite in the planar limit. Having realized the β-deformation in the manner described in the previous section the proof of finiteness of [4] can be repeated step by step in the present case.

The general philosophy behind our approach is as follows.

- The superficial degree of divergence of all planar supergraphs can be shown to be zero using a version of the power counting methods of [23], adapted to our formalism. This result assumes that all momenta in a supergraph contribute to the loop integral and provides a preliminary estimate.

- We then distinguish between internal and external momenta and focus on vertices attached to external legs. Using manipulations of the chiral derivatives we show that the superficial degree of divergence can be reduced to a negative value.

- The above analysis applies to the entire supergraph. The same analysis can be applied to prove that all subgraphs also have negative superficial degree of divergence.

- Having shown that all supergraphs and their subgraphs in the planar approximation have negative superficial degree of divergence, finiteness of all Green functions follows from Weinberg's theorem [24].

4.1 Supergraph power counting

In this subsection we explain how the superficial degree of divergence is estimated. A general procedure for determining the degree of divergence of diagrams in superspace was developed in [23]. These power counting rules were applied to $\mathcal{N} = 4$ Yang-Mills in light-cone superspace in [4] to show that all supergraphs are at most logarithmically divergent if all momenta contribute to the loop integrals. This result remains valid in the β-deformed theory at the planar level, whereas the same analysis provides a less stringent bound on the divergence of non-planar diagrams.

The first step in the analysis of the degree of divergence is to perform the fermionic integrals using the δ-functions in the propagator (3.17). This procedure is easily repeated for all the θ-integrals within a given loop until θ integrations at only two superspace points remain. The last two integrals are evaluated using the formula [4, 23]

$$\delta^8(\theta_{(1)} - \theta_{(2)}) \, d^4_{(1)} \, \bar{d}^4_{(1)} \, \delta^8(\theta_{(1)} - \theta_{(2)}) = \delta^8(\theta_{(1)} - \theta_{(2)}) \,, \tag{4.1}$$

which shows that when acting with the operator $d^4_{(1)} \bar{d}^4_{(1)}$ on the second δ-function only the fermionic derivatives contribute. This implies that (4.1), which could have potentially contributed four powers of momentum to the loop integral, instead has a null contribution. Note that any other combination of the chiral derivatives, involving less than four d's and four $\bar{d}$'s, is zero because it necessarily involves factors of $(\theta_{(1)} - \theta_{(1)})$. This result combined with the usual power counting rules, implies that the superficial degree of divergence of a generic supergraph in $\mathcal{N} = 4$ Yang-Mills is zero [3, 4].

This analysis is in general affected by the β-deformation. The effect of the modification described in section 3.2.2 is to introduce $\star$-products into supergraphs. In the case of planar supergraphs the step-wise fermionic integration leads to expressions of the form

$$\delta^8(\theta_{(1)} - \theta_{(2)}) \left[A(\theta_{(1)}) \star_{(1)} d^4_{(1)} \bar{d}^4_{(1)} \delta^8(\theta_{(1)} - \theta_{(2)}) \star_{(2)} B(\theta_{(2)}) \right] , \tag{4.2}$$

where $A(\theta_1)$ and $B(\theta_2)$ are arbitrary superfields and $\star_{(1)}$, $\star_{(2)}$ act at superspace points 1, 2. The presence of $\star$-products introduces phases implying that the expansion of the expression in brackets contains factors such as

$$(\mathrm{e}^{i\pi p_1} \theta_{(1)} - \mathrm{e}^{i\pi p_2} \theta_{(1)}) \,. \tag{4.3}$$

In planar diagrams, charge conservation ensures that $p_1 = p_2$, so we have a formula analogous to (4.1)

$$\delta^8(\theta_{(1)} - \theta_{(2)}) \left[A(\theta_{(1)}) \star_{(1)} d^4_{(1)} \bar{d}^4_{(1)} \delta^8(\theta_{(1)} - \theta_{(2)}) \star_{(2)} B(\theta_{(2)}) \right] = A(\theta_{(1)}) B(\theta_{(2)}) \delta^8(\theta_{(1)} - \theta_{(2)}), \tag{4.4}$$

which leads to the same conclusion concerning the degree of divergence of supergraphs. Despite the presence of non-trivial phase factors, combinations that involve less than eight chiral derivatives acting on a δ-function vanish. More details regarding planar supergraph power counting in the presence of $\star$-products are presented in appendix C.

In the case of non-planar supergraphs, the modification to the rules due to the $\star$-products is more complicated. In particular, the chiral derivatives can contribute extra factors of momentum to the loop integrals. Thus, the methods described here only offer a very poor upper bound on the degree of divergence of non-planar graphs.
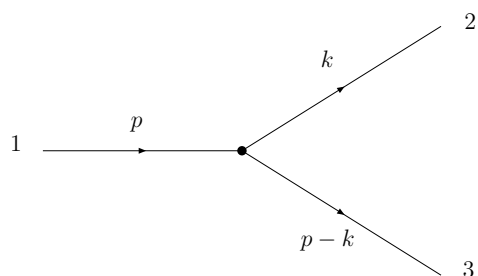
4.2 Analysis of planar n-point graphs

We now explicitly show how to implement the points listed at the beginning of section 4. As explained in the previous pages, the superficial degree of divergence of a generic planar supergraph is zero if all momenta contribute to the loop integrals. However, since the external legs in a supergraph do not contribute to these integrals, certain Wick contractions can potentially give rise to diagrams with positive degree of divergence. We analyze the contribution of different Wick contractions to generic supergraphs and, using the Feynman rules of the theory, we explain how to reduce, in each case, the degree of divergence from that determined by power counting to a negative value.

Notice that this analysis is applicable separately to both planar and non-planar diagrams, but it only allows us to conclude that the planar supergraphs are finite because for these we have a stronger bound on the superficial degree of divergence. A limited number of planar diagrams require special attention and these are discussed in subsection 4.2.3.

4.2.1 Graphs involving a cubic vertex

We first examine the case where an external leg is attached to a cubic vertex. We consider

$$\langle \Phi(z_1)\Phi(z_2)\Phi(z_3) \int \mathrm{d}^{12}z \, i\mathcal{L}_3(z)\rangle, \tag{4.5}$$

where

$$i\,\mathcal{L}_3 = (-g)\,\mathrm{Tr}\left[\frac{1}{12}\,\frac{1}{\partial_-}\,\Phi\,[\frac{\bar{d}^4}{\partial_-^2}\Phi, \partial\frac{\bar{d}^4}{\partial_-^2}\Phi]_\star + 4\,\frac{\bar{d}^4}{\partial_-^3}\Phi\,[\Phi, \bar\partial\Phi]_\star\right]. \tag{4.6}$$

We start with the first term in (4.6) and show that all the terms it produces on Wick contraction are finite. Assume leg 1 is external while 2 and 3 are internal.

We then have the following Wick contractions

$$-\frac{g}{12}\,\mathrm{Tr}'\bigg\{\frac{1}{\partial_-}\Delta_1[\frac{\bar{d}^4}{\partial_-^2}\Delta_2, \partial\frac{\bar{d}^4}{\partial_-^2}\Delta_3]_\star + (2\leftrightarrow 3)$$

$$+ \frac{1}{\partial_-}\Delta_2[\frac{\bar{d}^4}{\partial_-^2}\Delta_3, \partial\frac{\bar{d}^4}{\partial_-^2}\Delta_1]_\star + (2\leftrightarrow 3)$$

$$+ \frac{1}{\partial_-}\Delta_2[\frac{\bar{d}^4}{\partial_-^2}\Delta_1, \partial\frac{\bar{d}^4}{\partial_-^2}\Delta_3]_\star + \frac{1}{\partial_-}\Delta_3[\frac{\bar{d}^4}{\partial_-^2}\Delta_1, \partial\frac{\bar{d}^4}{\partial_-^2}\Delta_2]_\star\bigg\}. \tag{4.7}$$

The primed trace, which indicates that only the indices associated with the interaction point are summed over, is further explained in appendix B. For convenience, we will only explicitly write this trace in the first of a series of steps. In the following we will repeatedly integrate by parts the chiral derivatives that appear in the contractions above. The rules governing these manipulations are described in appendix A.2.

In the first term in (4.7), the presence of a $\frac{1}{\partial_-}$ acting on an external leg implies that we lose a factor of momentum from the denominator of the loop-integral making it potentially linearly divergent. In the first line, we integrate the $\bar{d}^4$ from leg 2 for example moving it to leg 1 (it cannot move to leg 3 since $\bar{d}^5 = 0$). This takes two powers of momentum out of the loop-integral, rendering it finite. The second line in momentum-space is

$$\frac{1}{k_-} \frac{1}{(p_- - k_-)^2} \frac{p}{p_-^2} \, \Delta_2[\bar{d}^4 \Delta_3, \bar{d}^4 \Delta_1]_\star \,. \tag{4.8}$$

The presence of the factor $p \frac{\bar{d}^4}{p_-^2}$ on the external leg improves the convergence of the integral by a single power of momentum. Hence this contribution is finite. The third line in (4.7) is more subtle. We start with the first term and integrate the superspace chiral derivatives from leg 3 to leg 2

$$\frac{1}{\partial_-} \Delta_2 [\frac{\bar{d}^4}{\partial_-^2} \Delta_1, \partial \frac{\bar{d}^4}{\partial_-^2} \Delta_3]_\star = \frac{\bar{d}^4}{\partial_-} \Delta_2 [\frac{\bar{d}^4}{\partial_-^2} \Delta_1, \partial \frac{1}{\partial_-^2} \Delta_3]_\star \,. \tag{4.9}$$

Using the last relation in (A.1) we rewrite this as

$$-\partial \frac{1}{\partial_-^2} \Delta_3 [\frac{\bar{d}^4}{\partial_-^2} \Delta_1, \frac{\bar{d}^4}{\partial_-} \Delta_2]_\star \,. \tag{4.10}$$

The third line in (4.7) now reads

$$-\partial \frac{1}{\partial_-^2} \Delta_3 [\frac{\bar{d}^4}{\partial_-^2} \Delta_1, \frac{\bar{d}^4}{\partial_-} \Delta_2]_\star + \frac{1}{\partial_-} \Delta_3 [\frac{\bar{d}^4}{\partial_-^2} \Delta_1, \partial \frac{\bar{d}^4}{\partial_-^2} \Delta_2]_\star \,. \tag{4.11}$$
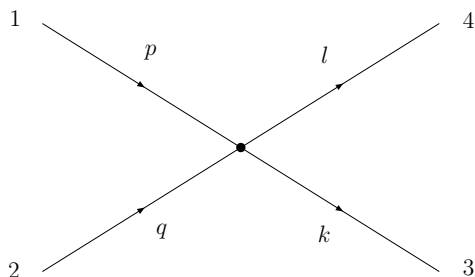
Working in momentum space, this becomes

$$-\left(\frac{p-k}{(p_- - k_-)^2} \frac{1}{p_-^2} \frac{1}{k_-} - \frac{1}{p_- - k_-} \frac{1}{p_-^2} \frac{k}{k_-^2} \right) \Delta_3 \, [\bar{d}^4 \Delta_1, \bar{d}^4 \Delta_2]_\star \,. \tag{4.12}$$

When tracking ultra-violet divergences, our focus is on large loop-momenta. For $k \gg p$, the leading term in parentheses vanishes implying finiteness of this contribution.

Having analyzed diagrams resulting from the first vertex in (4.6) we now move to the second vertex. The Wick contractions in this case yield

$$-4g \, \mathrm{Tr}' \Big\{ \frac{\bar{d}^4}{\partial_-^3} \Delta_1 [\Delta_2, \bar{\partial} \, \Delta_3]_\star + (2 \leftrightarrow 3)$$
$$+ \frac{\bar{d}^4}{\partial_-^3} \Delta_2 [\Delta_3, \bar{\partial} \, \Delta_1]_\star + (2 \leftrightarrow 3)$$
$$+ \frac{\bar{d}^4}{\partial_-^3} \Delta_3 [\Delta_1, \bar{\partial} \, \Delta_2]_\star + \frac{\bar{d}^4}{\partial_-^3} \Delta_2 [\Delta_1, \bar{\partial} \, \Delta_3]_\star \Big\} \,. \tag{4.13}$$

In line 1, both internal legs are free of $\bar{d}$'s. However, they are both attached to (internal) propagators that carry a factor of d^4 (see equation (3.17)). Integrating this factor of d^4 from either internal leg takes it out of the loop integral (since $d^5 = 0$) and ensures convergence. Line 2 is finite because the numerator involves factors of $\bar{p}$ (from $\bar{\partial}\Delta_1$), the external momentum, which factors out of the integral. As in the previous case, line 3 involves a little work. We start with the first term

$$\frac{\bar{d}^4}{\partial_-^3}\Delta_3[\Delta_1, \bar{\partial}\Delta_2]_\star = \frac{1}{\partial_-^3}\Delta_3[\Delta_1, \bar{\partial}\bar{d}^4\Delta_2]_\star \,, \tag{4.14}$$

which is rewritten, using antisymmetry of the $\star$-commutator and the last relation in (A.1), as

$$\bar{\partial}\bar{d}^4\Delta_2[\frac{1}{\partial_-^3}\Delta_3, \Delta_1]_\star = -\bar{\partial}\bar{d}^4\Delta_2[\Delta_1, \frac{1}{\partial_-^3}\Delta_3]_\star \,. \tag{4.15}$$

In momentum space, the entire third line is now

$$\left(\frac{\bar{p} - \bar{k}}{k_-^3} - \frac{\bar{k}}{(p_- - k_-)^3}\right)\bar{d}^4\Delta_2[\Delta_1, \Delta_3]_\star \,. \tag{4.16}$$

For $k \gg p$ the leading terms cancel and this makes the resulting integral finite.

4.2.2 Graphs involving a quartic vertex

We start from

$$\langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4) \int \mathrm{d}^{12}z\, i\mathcal{L}_4(z)\rangle \,, \tag{4.17}$$

where

$$i\,\mathcal{L}_4(z) = i\,\frac{g^2}{16}\,\mathrm{Tr}\bigg\{\frac{1}{\partial_-}[\Phi, \partial_-\Phi]\frac{1}{\partial_-}\bigg[\frac{\bar{d}^4}{\partial_-^2}\Phi, \frac{\bar{d}^4}{\partial_-}\Phi\bigg]_\star + \frac{1}{2}\bigg[\Phi, \frac{\bar{d}^4}{\partial_-^2}\Phi\bigg]_\star\bigg[\Phi, \frac{\bar{d}^4}{\partial_-^2}\Phi\bigg]_\star\bigg\} \,. \tag{4.18}$$

We will analyze separately the cases in which either one or two legs of the quartic vertex are external. In the following, we will ignore the overall factor of $i\frac{g^2}{16}$.

Graphs involving two external lines. We choose legs 1 and 2 to be external while 3 and 4 are internal.

The first quartic vertex in (4.18) gives rise to twenty-four contractions. The analysis of the majority of these involves the same manipulations utilized in the case of the cubic vertex: superspace chiral derivatives, d's or $\bar{d}$'s, are integrated by parts from an internal leg onto an external one to remove factors of momentum from the numerator of loop integrals. The only noticeable difference with respect to the analysis in the previous section is that the integrations by parts can produce non-trivial phase factors which, however, do not affect the ultra-violet behavior of the diagrams. For completeness we discuss these contractions in appendix E, focussing instead here on those which require special attention. These are

$$\mathrm{Tr}'\left\{\frac{1}{\partial_-}[\Delta_1, \partial_- \Delta_3]_\star \frac{1}{\partial_-}\left[\frac{\bar{d}^4}{\partial_-^2}\Delta_2, \frac{\bar{d}^4}{\partial_-}\Delta_4\right]_\star + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4)\right\}. \quad (4.19)$$

In terms of momenta, the first term becomes

$$\frac{k_-}{k_- + p_-}\frac{1}{q_- + l_-}\frac{1}{q_-^2}\frac{1}{l_-}[\Delta_1, \Delta_3]_\star[\bar{d}^4\Delta_2, \bar{d}^4\Delta_4]_\star. \quad (4.20)$$

The potentially divergent contribution in this expression is cancelled by contributions from the second quartic vertex in (4.18). The twenty-four contractions from the second quartic vertex reduce to twelve terms due to the symmetry in the expression. This symmetry factor cancels the $\frac{1}{2}$ in front of the vertex. The contractions that cancel the divergences in (4.19) can be written as

$$\left[\Delta_1, \frac{\bar{d}^4}{\partial_-^2}\Delta_3\right]_\star\left[\Delta_4, \frac{\bar{d}^4}{\partial_-^2}\Delta_2\right]_\star + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4). \quad (4.21)$$

We integrate the $\bar{d}^4$ from leg 3 to leg 4 (if a single $\bar{d}$ hits the external leg 1, the integral is rendered finite). In momentum space, the first term is

$$\frac{1}{k_-^2}\frac{1}{q_-^2}[\Delta_1, \Delta_3]_\star[\bar{d}^4\Delta_4, \bar{d}^4\Delta_2]_\star. \quad (4.22)$$

In the large loop-momentum limit $k, l \gg p, q$ the leading order term exactly cancels against that in (4.20) implying that the combined contribution is finite. In appendix E we present the finiteness analysis for the remaining contractions involving the second quartic vertex.

Graphs involving one external line. We choose leg 1 to be external keeping 2, 3 and 4 internal.

We split our analysis into two portions. We consider first graphs in which the external leg 1 does not have a factor of $\bar{d}^4$ on it. In this case, the contractions for the first quartic vertex in (4.18) are

$$\mathrm{Tr}'\bigg\{\frac{1}{\partial_-}[\Delta_1, \partial_- \Delta_2]_\star \frac{1}{\partial_-}\left[\frac{\bar{d}^4}{\partial_-^2}\Delta_3, \frac{\bar{d}^4}{\partial_-}\Delta_4\right]_\star + \frac{1}{\partial_-}[\Delta_2, \partial_- \Delta_1]_\star \frac{1}{\partial_-}\left[\frac{\bar{d}^4}{\partial_-^2}\Delta_3, \frac{\bar{d}^4}{\partial_-}\Delta_4\right]_\star$$

$$+ \frac{1}{\partial_-}[\Delta_1, \partial_- \Delta_2]_\star \frac{1}{\partial_-}\left[\frac{\bar{d}^4}{\partial_-^2}\Delta_4, \frac{\bar{d}^4}{\partial_-}\Delta_3\right]_\star + \frac{1}{\partial_-}[\Delta_2, \partial_- \Delta_1]_\star \frac{1}{\partial_-}\left[\frac{\bar{d}^4}{\partial_-^2}\Delta_4, \frac{\bar{d}^4}{\partial_-}\Delta_3\right]_\star$$

$$+ (\text{permutations of } 2, 3, 4)\bigg\}. \quad (4.23)$$
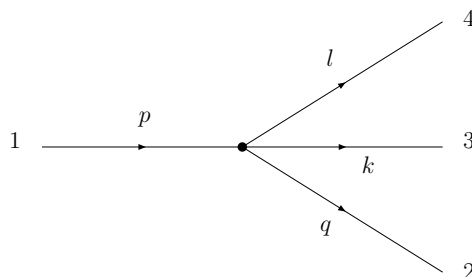
The contractions from the second quartic vertex in (4.18) read

$$\left[\Delta_1, \frac{1}{\partial_-^2}\Delta_2\right]_\star \left[\bar{d}^4\Delta_3, \frac{\bar{d}^4}{\partial_-^2}\Delta_4\right]_\star + \left[\Delta_1, \frac{1}{\partial_-^2}\Delta_2\right]_\star \left[\bar{d}^4\Delta_4, \frac{\bar{d}^4}{\partial_-^2}\Delta_3\right]_\star + (\text{permutations of } 2, 3, 4)\,, \tag{4.24}$$

where we have integrated the $\bar{d}^4$ away from leg 2 to legs 3 and 4 (if they move to leg 1, the integral is finite). In momentum space, the sum of equations (4.23) and (4.24) is proportional to

$$\frac{(p_- + k_-)(l_- + q_-)^2(l_- - q_-) + q_- k_-^3 - q_- k_-^2 p_- - k_-^3 l_- + l_- k_-^2 p_-}{(p_- + k_-)(l_- + q_-)l_-^2 q_-^2 k_-^2}\,. \tag{4.25}$$

Using momentum conservation, in the large loop-momentum limit, the dominant terms in this expression behave as

$$\frac{(k_- + 2l_-)p_-^2}{k_-^3 l_-^2 (k_- + l_-)^2}\,, \tag{4.26}$$

implying that the graph is finite.

The final case is when the external leg has a factor of $\bar{d}^4$ on it. When studying a complicated supergraph, if we locate a single external leg (free of $\bar{d}^4$) attached to a three or four-point vertex, finiteness follows based on the arguments presented so far. Thus the only cause for concern is a supergraph which has factors of $\bar{d}^4$ on all its external legs. If this is the case, then in order for the expression to survive the integration over the entire measure $\int d^4\theta d^4\bar{\theta}$ we necessarily need a factor of d^4 to compensate the $\bar{d}^4$. This factor of d^4 must come from the internal structure of the supergraph and ensures that the integrals are rendered finite.

4.2.3 Exceptional cases

There are a few planar supergraphs that require special attention. These are depicted in figure 1.

The analysis of section 4 is insufficient to prove finiteness for some specific contractions corresponding to diagrams of the types (1), (2) and (3) in figure 1. The chiral derivative structure in these contractions is explicitly shown in the figures. Notice that these graphs have factors of $\bar{d}^4$ acting on all the external legs and thus they belong to the class discussed at the end of the previous subsection. Our proof assumes that factors of d^4 can always be
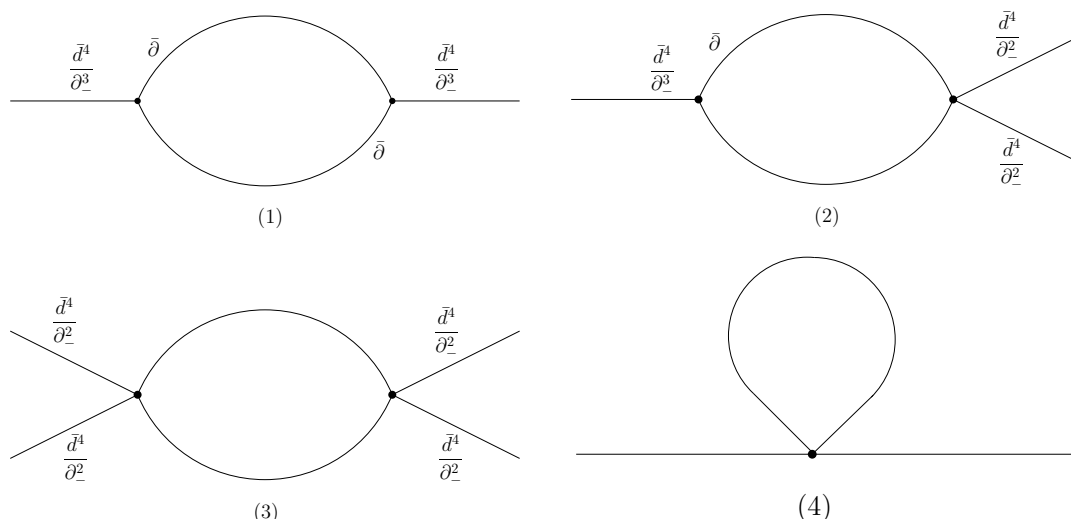
Figure 1: Diagrams to be treated separately.

integrated by parts from internal to external lines for diagrams of the type in figure 1. In the present cases, however, after the first d^4 has been moved out of the loop, the second factor of d^4 can also move to the other internal line which now has no d's acting on it. The degree of divergence at this stage is still logarithmic. The resolution to this comes from the chiral derivative structure. The θ-integrals in all these graphs trivially vanish in the planar limit since there are not enough chiral derivatives acting on the δ-functions.

The diagram (4) in figure 1 involves a self-contraction which is also not dealt with in our arguments of the previous section. In addition, there are subtleties associated with defining this graph in superspace in the presence of $\star$-products. Resorting to a simple component calculation, however, one can verify that in the planar limit the one-loop two-point function of the β-deformed theory is identical to that in $\mathcal{N} = 4$ Yang-Mills. Thus the one-loop two-point function which includes the contribution (4) is finite and has the correct asymptotic behavior at large momentum.

We have thus shown that all supergraphs have a negative superficial degree of divergence. This analysis applies equally well to all subgraphs within a given supergraph. This allows us to use Weinberg's theorem [24] to conclude that all the Green functions of the theory are finite. Notice that Weinberg's theorem in its original form requires Euclidean signature and therefore there are potential subtleties when using it in light-cone gauge. In our formalism, the Wick rotation into Euclidean space is permitted thanks to the residual gauge freedom [25], which allows us to choose the pole structure $(p_- + i\epsilon p_+)^{-1}$ for the operator $\frac{1}{\partial_-}$ [3]. We also point out that Weinberg's theorem has been generalized to Lorentzian signature in [26].

5. Conclusions

In this paper we have studied the finiteness properties of a special example of β-deformed

$\mathcal{N} = 4$ Yang-Mills theory, involving a single real deformation parameter. Theories in this class, despite the reduced amount of supersymmetry, preserve many of the remarkable properties of the parent $\mathcal{N} = 4$ theory. Our methods show that this particular β-deformed $\mathcal{N} = 4$ SYM is conformally invariant in the planar limit. The essential ingredient of this analysis was the realization that the deformed theory, despite having only $\mathcal{N} = 1$ supersymmetry, could still be formulated in $\mathcal{N} = 4$ light-cone superspace using suitably defined superspace $\star$-products. In this formulation the deformation preserves the ultra-violet behavior of the $\mathcal{N} = 4$ theory, in the planar limit, thanks to the properties of the superspace $\star$-product, which allowed us to prove the finiteness of all the Green functions in the theory following the same steps previously utilized in [3, 4] in the $\mathcal{N} = 4$ case.

The results in this paper are valid to all orders in planar perturbation theory. Instanton effects, which have been studied in the β-deformed theory in [27], generalizing previous work done in $\mathcal{N} = 4$ SYM [28], are exponentially suppressed in the planar approximation and therefore cannot spoil the conformal invariance of the theory in this limit. However, one of the remarkable features of the β-deformed theory is that it inherits from $\mathcal{N} = 4$ SYM a modified form of S-duality [29]. Instantons are expected to play a crucial role in the realization of this symmetry.

The proof presented here is valid for arbitrary choice of the gauge group, but the case of SU(N) is of special interest in the context of the AdS/CFT correspondence. The β-deformation studied in this paper is believed to be dual to the supergravity background constructed in [10]. Our arguments, showing that the deformed SYM theory is conformally invariant to all orders in perturbation theory in the planar limit, suggest that the SO(4,2) isometries of the supergravity solution of [10] should not be affected by string tree-level corrections.

Our analysis opens up many venues for generalizations. A straightforward extension involves further deforming the theory with the addition of mass terms for the fields in the $\mathcal{N} = 1$ chiral multiplets. These mass deformations can preserve $\mathcal{N} = 1$ supersymmetry or break it completely, but do not spoil the ultra-violet finiteness of the theory, although they obviously break conformal invariance.

As already mentioned the theory considered in this paper belongs to the class of deformations of $\mathcal{N} = 4$ SYM characterized by the superpotential (1.1). It has been argued [6] that the generic theory in this family can be rendered finite by imposing a single relation among the parameters,

$$\gamma(g, h, \beta) = 0 \,. \tag{5.1}$$

Our results show that in the planar approximation the theory involving only a real deformation parameter, β, is finite without any conditions on β. This is in agreement with the explicit results of [11–13]. Moreover it implies that the general condition for finiteness (5.1), specialized to the case of a single real β parameter and $h = 1$, should be identically satisfied in the large N limit at all orders in the Yang-Mills coupling. We note that a different argument for the all-order finiteness of this theory in the planar limit was provided in [13].

Although we have only discussed the case of real β, the light-cone superspace formalism

is well suited to study the case where the deformation parameter is made complex. In this case, even in the planar limit, the condition (5.1) may remain non-trivial. Therefore our approach should provide interesting insights into the surface of finite theories defined by (5.1).

We also believe that the analysis that we presented can be generalized to the case of a β-deformation that breaks all the supersymmetries in the theory [30]. This would be extremely interesting because we would then have a proof of conformal invariance, in the planar limit, for a non-supersymmetric field theory. These issues are currently under investigation.

Having shown that the β-deformed theory preserves conformal invariance, it is natural to study the spectrum of scaling dimensions of gauge-invariant operators in this model. Various results have been presented in [11–13, 31], confirming and extending the earlier analysis of [8, 32]. The problem of computing the spectrum of scaling dimensions can be efficiently recast as an eigenvalue problem for the dilatation operator of the theory. In the planar limit on which we have focussed, in the case of the $\mathcal{N} = 4$ Yang-Mills theory, the dilatation operator can be related to the Hamiltonian of an integrable spin chain allowing for the use of powerful techniques such as the Bethe Ansatz to compute anomalous dimensions [33]. It may be interesting to generalize some of these techniques for use in studying the β-deformed theory.

Acknowledgments

A. Properties of star products

We present here relevant relations satisfied by the star products.

A.1 Properties of the component ∗-product

The ∗-product satisfies the following properties

$$
\begin{aligned}
&A * (B * C) = (A * B) * C \qquad \text{(associativity)} \\
&[A, [B, C]_*]_* + [B, [C, A]_*]_* + [C, [A, B]_*]_* = 0 \qquad \text{(Jacobi identity)} \\
&A * B = AB \qquad \text{(for } Q_A \cdot Q_B = 0 \text{ or } Q_A + Q_B = 0) \\
&\text{Tr}(A[B, C]_*) = \text{Tr}(B[C, A]_*) = \text{Tr}(C[A, B]_*) \qquad \text{(for } Q_A + Q_B + Q_C = 0) \,.
\end{aligned}
\tag{A.1}
$$

A.2 Properties of the superspace $\star$-product

The superspace $\star$ is essentially a superspace realization of the component $*$. Naturally it also satisfies all the properties listed above. The presence of $\star$-products in superspace expressions modifies the rules for partial integration of chiral derivatives. In this appendix, we describe manipulations of chiral derivatives. In the following, the index on the chiral derivatives refers to their flavor.[3]

In general, the integration by parts of chiral derivatives in superspace expressions involving $\star$-products gives rise to phase factors. This is a consequence of the modification of the standard Leibniz rule, which, in the presence of $\star$-products, becomes

$$d^1(F \star G) = d^1 F \star (\mathrm{e}^{-i\pi\beta\, Q^1} G) + (\mathrm{e}^{i\pi\beta\, Q^1} F) \star d^1 G \,. \tag{A.2}$$

Similar relations hold for the other chiral derivatives. In this relation, Q^1 is an operator, which acts differently on the various terms in the expansion of the superfields F and G. Therefore, in order to prove (A.2), it is convenient to decompose the superfields into pieces which have definite flavor charge. This is achieved by the standard θ-expansion

$$
\begin{aligned}
F &= f_{(0,0)} + f_{(1,0)m}\theta^m + f_{(0,1)}^m \bar\theta_m + \cdots \\
G &= g_{(0,0)} + g_{(1,0)m}\theta^m + g_{(0,1)}^m \bar\theta_m + \cdots \,.
\end{aligned} \tag{A.3}
$$

We then have

$$d^1 (f \star g) = d^1 \left[f\, g\, e^{i\pi\beta(Q_f^1 Q_g^2 - Q_f^2 Q_g^1)} \right], \tag{A.4}$$

where the terms in the exponential are now numbers and f and g are generic terms in the expansions (A.3). In (A.4) we can now use the ordinary Leibniz rule to obtain

$$d^1 (f \star g) = (d^1 f)\, g\, e^{i\pi\beta(Q_f^1 Q_g^2 - Q_f^2 Q_g^1)} + f\,(d^1 g)\, e^{i\pi\beta(Q_f^1 Q_g^2 - Q_f^2 Q_g^1)} \,. \tag{A.5}$$

From the definition of the supercharges in (3.11) and table 2 it follows that

$$Q_{d^1 f}^1 = Q_f^1 \quad \text{and} \quad Q_{d^1 f}^2 = Q_f^2 + 1 \,, \tag{A.6}$$

implying that (A.5) can be rewritten as

$$d^1 (f \star g) = (d^1 f) \star g\, e^{-i\pi\beta\, Q_g^1} + f \star (d^1 g)\, e^{i\pi\beta\, Q_f^1} \,. \tag{A.7}$$

We now re-sum the component pieces into superfields and this yields (A.2).

In manipulating superspace expressions it is often necessary to integrate by parts multiple chiral derivatives. This is easily achieved by repeatedly using (A.2). In particular, for

$$d^1 d^2 d^3 d^4 (F \star G), \tag{A.8}$$

---

[3]This notation should not be confused with that in the main text where d^4 was used to denote the product of all four chiral derivatives.

we obtain

$$d^1 d^2 d^3 d^4 (F \star G) = (d^1 d^2 d^3 d^4 F) \star G - \mathrm{e}^{p_1} (d^2 d^3 d^4 F) \star d^1 G + \mathrm{e}^{p_2} (d^3 d^4 F) \star d^2 d^1 G$$
$$+ \cdots + F \star d^1 d^2 d^3 d^4 G \,. \tag{A.9}$$

The exact values of the phase factors, e^{p_1}, e^{p_2}, ..., which can be computed using (A.2) are not essential to our analysis. This is because our proof, in section 4, relies on moving four chiral derivatives which does not produce a phase. That is, partially integrating four chiral derivatives, ensures that the complicated phase factors cancel each other. This is because the product of all four chiral derivatives is uncharged under the U(1)×U(1) flavor symmetry. In particular, for three generic superfields F, G and H, the Leibniz rule implies that

$$\int \mathrm{d}^{12}z \, \{(d^1 d^2 d^3 d^4 F) \star G\} \, d^1 d^2 d^3 d^4 H = \int \mathrm{d}^{12}z \, \{F \star (d^1 d^2 d^3 d^4 G)\} \, d^1 d^2 d^3 d^4 H \,, \tag{A.10}$$

since the chiral derivatives cannot move from F to H. Equation (A.10) is non-trivial to prove directly in superspace because the superfields do not carry definite U(1)×U(1) charge. However, the result is straightforward to verify, by decomposing the (generic) superfields into pieces which carry definite flavor charge, as was done in the derivation of the modified Leibniz rule (A.2).

B. Wick contraction notation

The superfield propagator is

$$\langle (\Phi)_v^u(z_1) \, (\Phi)_s^r(z_2) \rangle = \langle \Phi^a \, (T^a)_v^u \Phi^b \, (T^b)_s^r \rangle = \Delta_{vs}^{ur}(z_1 - z_2) \,, \tag{B.1}$$

A sample Wick contraction is

$$\left(\Phi_{v_1}^{u_1}(z_1) \, \Phi_{v_2}^{u_2}(z_2) \, \Phi_{v_3}^{u_3}(z_3) \, \Phi_{v_4}^{u_4}(z_4) \right) \left(\left[\Phi(z), \frac{\bar{d}^4}{\partial_-^2} \Phi(z) \right]_{\star} \right)_s^r \left(\left[\Phi(z), \; \frac{\bar{d}^4}{\partial_-^2} \Phi(z) \right]_{\star} \right)_r^s , \tag{B.2}$$

where

$$\left(\left[\Phi(z), \frac{\bar{d}^4}{\partial_-^2} \Phi(z) \right]_{\star} \right)_s^r = \Phi_m^r(z) \star \frac{\bar{d}^4}{\partial_-^2} \Phi_s^m(z) - \frac{\bar{d}^4}{\partial_-^2} \Phi_m^r(z) \star \Phi_s^m(z) \,. \tag{B.3}$$

We treat the propagator between superspace points z and z_1 as a matrix in the indices associated with the point z. We simplify our notation considerably by not explicitly showing the z_1 indices and the dependence on $(z - z_1)$. We write

$$\Delta_{r \, v_1}^{s \, u_1}(z - z_1) \equiv (\Delta_1)_r^s \equiv \Delta_1 \,. \tag{B.4}$$

In this new notation, the contraction reads

$$\mathrm{Tr}' \, \{ [\Delta_1, \Delta_2]_{\star} \, [\Delta_3, \Delta_4]_{\star} \} \,, \tag{B.5}$$

where the symbol Tr' refers to the fact that only the indices associated with the point z are contracted.
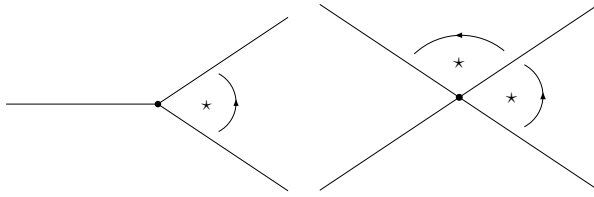
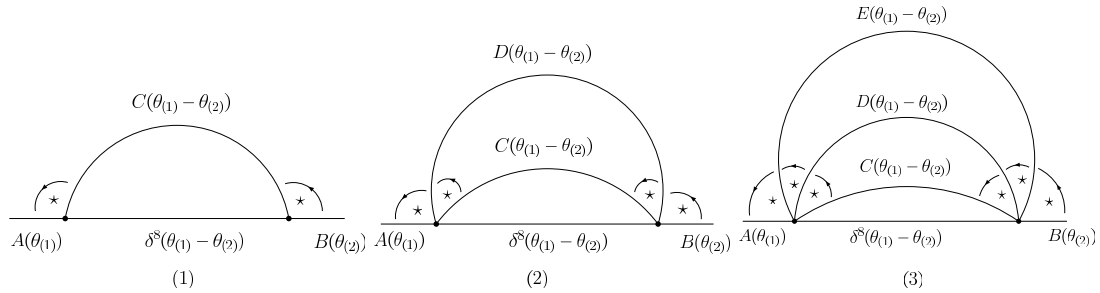**Figure 2:** Star-products in supergraphs.

Figure 3: Loops in θ-space produced by the step-wise fermionic integrations.

C. Planar supergraphs and power counting

This appendix illustrates some basic manipulations of planar supergraphs in the presence of $\star$-products. The $\star$-products acting at three- and four-point vertices are shown explicitly in figure 2. The arrows on the $\star$'s between two lines refer to the order in which the corresponding superfields are multiplied.

Planar supergraphs in the β-deformed theory are characterized by the fact that all these $\star$-products that act between adjacent legs in a vertex have the same orientation. In other words the $\star$'s all act either clockwise or counter-clockwise. Using the properties listed in appendix A these $\star$'s can be moved around the vertex as long as their orientation is preserved. This property is used in the procedure of step-wise integration over the fermionic variables as explained in section 4.1.

The procedure of step-wise integration over the θ's using the δ-functions in the superfield propagators shrinks internal lines in a supergraph and can result in self-contracting vertices. These pose a potential problem since their correct definition in the presence of $\star$-products is rather subtle. However, this type of vertex can be avoided by carefully choosing the order in which the θ-integrals are performed. It is easy to verify that self-contractions only arise when shrinking the internal lines in graphs of the type shown in figure 3. Therefore we explain below how to treat generic graphs in this class. Notice that self-contracting lines can also be induced by the Feynman rules before the shrinking process is initiated. These primitive self-contracting vertices are treated as explained in subsection 4.2.3.

In all these diagrams, the approach is the same. We illustrate the method in the case of the diagram (2) in the figure (note that the diagram (1) is simply dealt with by using
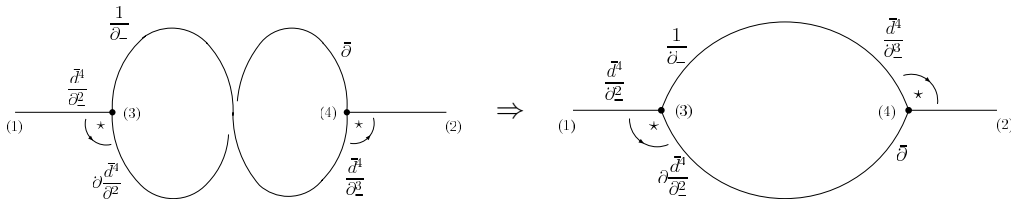
Figure 4: Non-planar contribution to the one-loop two-point function.

formula (4.4)). Using the fact that the $\star$-product is associative, we organize the order of $\star$-products of fields so that at point 1 we have $C\star(D\star A)$ and at point 2 $(B\star D)\star C$. We now θ-expand legs A, B and D into terms of definite charge under $U(1)\times U(1)$. The charges carried by the incoming and outgoing legs A and B are equal and opposite. Applying charge conservation along a given internal line, for example C, we see that for each $SU(4)$ flavor, the variables $\theta_{(1)}$ and $\theta_{(2)}$ pick up equal phases. It is clear that such a procedure applies to more complicated cases such as (3) in figure 3.

When dealing with graphs that have more external legs, it is useful to view these external legs as a single block when applying the analysis described above. It is important to note that this "block" usually contains $\star$'s in it and hence has a non-trivial dependence on β.

In some cases, manipulations on supergraphs can lead to an uneven (non-singlet) combination of the chiral derivatives. In these situations it might appear that charge conservation previously used may be violated. However these uneven distributions of the chiral derivatives always appear from partial integrations and hence the resulting vertices are always accompanied by phase factors as explained in appendix A.2. These phase factors cancel against those from the apparent violation of charge conservation as a consequence of the modified Leibniz rule (A.2). This implies that in similar situations the $\star$-products can be evaluated assuming that the $\star$ does not act on the d's. This simple rule is valid for any effective vertex arising from our procedure. The methods applied to graphs having an even distribution of chiral derivatives therefore are also applicable here.

D. Planar versus non-planar supergraphs

In this appendix, we explicitly illustrate the difference between planar and non-planar graphs in the context of power counting. This difference, best illustrated with the two-point function, is explained in the case of a specific Wick contraction.

Our starting point is the non-planar graph shown in figure 4. We will explain why the power counting rules described in subsection 4.1 and in appendix C are less useful in this case. Having done this, we turn to the planar case and show why the same power counting rules work in that case exactly as with $\mathcal{N}=4$ Yang-Mills.

Reading off Feynman rules from the action (3.12), we see that figure 4 is proportional to[4]

$$\int \mathrm{d}^4\theta_{(3)} \, \mathrm{d}^4\bar{\theta}_{(3)} \, \mathrm{d}^4\theta_{(4)} \, \mathrm{d}^4\bar{\theta}_{(4)} \, \mathrm{d}^4k \, \frac{(p-k)(\bar{p}-\bar{k})}{p_\mu^4 p_-^2 k_\nu^2 (p_- - k_-)^2 (p_\rho - k_\rho)^2} \, \delta^8(\theta_{(3)} - \theta_{(4)}) \qquad \text{(D.1)}$$
$$\times [d_{(1)}]^4 [\bar{d}_{(1)}]^4 \delta^8(\theta_{(1)} - \theta_{(3)}) \star_3 [\bar{d}_{(3)}]^4 [d_{(3)}]^4 \delta^8(\theta_{(3)} - \theta_{(4)}) \star_4^{-1} [d_{(4)}]^4 \delta^8(\theta_{(4)} - \theta_{(2)}) \,,$$

where the $\star^{-1}$ operation is simply defined by

$$F \star^{-1} G = G \star F \,, \qquad \text{(D.2)}$$

where F and G represent superfields or products of superfields.

We will explain the effect of the $\star$-deformation by considering the contribution of this graph to the $\langle \bar{\chi}_1 \chi^1 \rangle$ two-point function. For this contribution, we need to project the two external legs in the supergraph onto the corresponding fermion components. At external leg 1, this is achieved by acting with $\bar{d}_{(1)1}$ and then setting $\theta_{(1)}^m = \bar{\theta}_{(1)m} = 0$. This yields

$$\bar{d}_{(1)1} [d_{(1)}]^4 [\bar{d}_{(1)}]^4 \delta^8(\theta_{(1)} - \theta_{(3)}) \big|_{\theta_{(1)} = \bar{\theta}_{(1)} = 0} = \sqrt{2} p_- \bar{\theta}_{(3)1} \prod_{m=2}^4 \left[-1 - \frac{p_-}{\sqrt{2}} \theta_{(3)}^m \bar{\theta}_{(3)m} \right]. \quad \text{(D.3)}$$

At external leg 2, the projection onto χ^1 requires that we act with the operator $\bar{d}_{(2)2} \bar{d}_{(2)3} \bar{d}_{(2)4}$ and then set $\theta_{(2)}^n = \bar{\theta}_{(2)n} = 0$. This computation gives

$$\bar{d}_{(2)2} \bar{d}_{(2)3} \bar{d}_{(2)4} [d_{(4)}]^4 \delta^8(\theta_{(4)} - \theta_{(2)}) \big|_{\theta_{(2)} = \bar{\theta}_{(2)} = 0} = -\theta_{(4)}^1 \prod_{n=2}^4 \left[1 + \frac{p_-}{\sqrt{2}} \theta_{(4)}^n \bar{\theta}_{(4)n} \right]. \quad \text{(D.4)}$$

Having projected the two external legs onto the required fermionic states, we now expand the piece between the $\star$-products as

$$[\bar{d}_{(3)}]^4 [d_{(3)}]^4 \delta^8(\theta_{(3)} - \theta_{(4)}) = \prod_{p=1}^4 \left[-1 + \sqrt{2} k_- \theta_{(3)}^p \bar{\theta}_{(4)p} - \frac{1}{\sqrt{2}} k_- (\theta_{(3)}^p \bar{\theta}_{(3)p} + \theta_{(4)}^p \bar{\theta}_{(4)p}) \right.$$
$$\left. - \frac{1}{2} k_-^2 \theta_{(3)}^p \bar{\theta}_{(3)p} \theta_{(4)}^p \bar{\theta}_{(4)p} \right] \qquad \text{(D.5)}$$

The deformation will produce phase factors that we need to identify. The phase factors due to the $\star_3$ in (D.1) arise from the following term

$$\left\{ \bar{\theta}_{(3)1} \star_3 \prod_{p=1}^4 \theta_{(3)}^p \right\} \bar{\theta}_{(4)p} \,, \qquad \text{(D.6)}$$

while the phase factors from the $\star_4^{-1}$ are due to the term

$$\prod_{p=1}^4 \theta_{(3)}^p \{ \bar{\theta}_{(4)p} \star_4^{-1} \theta_{(4)}^1 \} \,. \qquad \text{(D.7)}$$

---

[4]Some of the equations in this appendix contain single chiral derivatives. To avoid ambiguities in the notation we therefore denote the product of four chiral or anti-chiral derivatives respectively by $[d_{(i)}]^4$ and $[\bar{d}_{(i)}]^4$, where the subscript (i) refers to the superspace point.
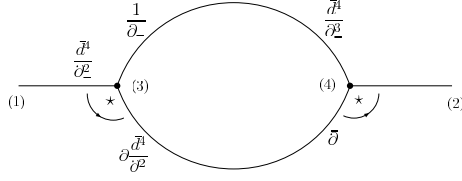
Figure 5: Planar contribution to the one-loop two-point function.

These phase factors are easy to compute using table 2. Once the $\star$-products have been evaluated, we are free to perform the integral over $\theta_{(4)}$ with the help of the first δ-function in (D.1). Since this sets $\theta_{(3)} = \theta_{(4)}$, we will no longer explicitly write the (3) index in what follows. Thus, the contribution of (D.1) to $\langle \bar{\chi}_1 \chi^1 \rangle$ is

$$
\int d^4\theta \, d^4\bar{\theta} \, d^4k \, \frac{(p-k)(\bar{p}-\bar{k})}{p_\mu^4 p_-^2 k_\nu^2 (p_- - k_-)^2 (p_\rho - k_\rho)^2} \, (-2p_-^2)[\theta^1\bar{\theta}_1][\theta^4\bar{\theta}_4]
$$
$$
\times [\sqrt{2}\theta^2\bar{\theta}_2\{p_-k_-(e^{i\pi\beta}e^{i\pi\beta}-1)\}][\sqrt{2}\theta^3\bar{\theta}_3\{p_-k_-(e^{-i\pi\beta}e^{-i\pi\beta}-1)\}] \,. \qquad \text{(D.8)}
$$

This step illustrates the effect of the β-deformation on non-planar supergraphs. The shrinking of lines in non-planar graphs is non-trivial due to the $\star(\ldots)\star^{-1}$ structure. This structure limits our ability to move around $\star$'s (to free up a δ-function) and is responsible for the phases, from the two $\star$'s, adding up and producing factors like $e^{-i\pi\beta}e^{-i\pi\beta}$ in (D.8).

We now perform the remaining θ-integration to obtain

$$
\int d^4k \, \frac{(p-k)(\bar{p}-\bar{k})}{p_\mu^4 p_-^2 k_\nu^2 (p_- - k_-)^2 (p_\rho - k_\rho)^2}
$$
$$
\times (-2p_-^2)[p_-^2 - 2p_-k_-(\cos 2\pi\beta - 1) - 2k_-^2(\cos 2\pi\beta - 1)] \,. \qquad \text{(D.9)}
$$

Although the first term is logarithmic, the second and third are linearly and quadratically divergent respectively. Thus the power counting procedure of subsection C only offers a poor upper bound on the superficial degree of divergence of this non-planar supergraph, namely $D = 2$. Thus the methods of section 4 (which ensured the cancellation of the logarithmic divergences in planar supergraphs) only prove the cancellation of quadratic divergences in non-planar supergraphs.

Having analyzed in detail the non-planar case we are in a position to easily understand why planar supergraphs are much easier to handle. The graph in figure 5 evaluates to

$$
\int d^4\theta_{(3)} \, d^4\bar{\theta}_{(3)} \, d^4\theta_{(4)} \, d^4\bar{\theta}_{(4)} \, d^4k \, \frac{(p-k)(\bar{p}-\bar{k})}{p_\mu^4 p_-^2 k_\nu^2 (p_- - k_-)^2 (p_\rho - k_\rho)^2} \, \delta^8(\theta_{(3)} - \theta_{(4)}) \qquad \text{(D.10)}
$$
$$
\times [d_{(1)}]^4 [\bar{d}_{(1)}]^4 \delta^8(\theta_{(1)} - \theta_{(3)}) \star_3 [\bar{d}_{(3)}]^4 [d_{(3)}]^4 \delta^8(\theta_{(3)} - \theta_{(4)}) \star_4 [d_{(4)}]^4 \delta^8(\theta_{(4)} - \theta_{(2)}) \,.
$$

We see immediately that the $\star$-structure differs from that in (D.1). This difference implies that the phase factors produced in (D.8), instead of adding up now cancel. Once again, we focus on the contribution of this two-point function to $\langle \bar{\chi}_1 \chi^1 \rangle$. Proceeding in exactly the

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same manner described so far, we find that this contribution is

$$\frac{-2(p-k)(\bar{p}-\bar{k})p_-^2}{p_\mu^4 k_\nu^2 (p_- - k_-)^2 (p_\rho - k_\rho)^2} \,, \tag{D.11}$$

which is logarithmically divergent. Since this graph has superficial degree of divergence equal to zero, our treatment of it as described in section 4 ensures that it is finite.

We remind the reader that in the planar limit, the one-loop two-point function of the β-deformed theory is identical to that in $\mathcal{N}=4$ Yang-Mills and has the correct asymptotic behavior at large momentum.

E. Quartic vertex contractions

E.1 Graphs involving two external legs

The twenty-four contractions induced by the first quartic vertex in (4.18) are

$$
\begin{aligned}
\mathrm{Tr}' \Bigg\{ & \frac{1}{\partial_-}[\Delta_3, \partial_- \Delta_4]_\star \frac{1}{\partial_-}\left[\frac{\bar{d}^4}{\partial_-^2}\Delta_1, \frac{\bar{d}^4}{\partial_-}\Delta_2\right]_\star + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4) \\
& + \frac{1}{\partial_-}[\Delta_1, \partial_- \Delta_2]_\star \frac{1}{\partial_-}\left[\frac{\bar{d}^4}{\partial_-^2}\Delta_3, \frac{\bar{d}^4}{\partial_-}\Delta_4\right]_\star + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4) \\
& + \frac{1}{\partial_-}[\Delta_3, \partial_- \Delta_1]_\star \frac{1}{\partial_-}\left[\frac{\bar{d}^4}{\partial_-^2}\Delta_4, \frac{\bar{d}^4}{\partial_-}\Delta_2\right]_\star + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4) \\
& + \frac{1}{\partial_-}[\Delta_3, \partial_- \Delta_1]_\star \frac{1}{\partial_-}\left[\frac{\bar{d}^4}{\partial_-^2}\Delta_2, \frac{\bar{d}^4}{\partial_-}\Delta_4\right]_\star + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4) \\
& + \frac{1}{\partial_-}[\Delta_2, \partial_- \Delta_3]_\star \frac{1}{\partial_-}\left[\frac{\bar{d}^4}{\partial_-^2}\Delta_4, \frac{\bar{d}^4}{\partial_-}\Delta_1\right]_\star + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4) \\
& + \frac{1}{\partial_-}[\Delta_1, \partial_- \Delta_3]_\star \frac{1}{\partial_-}\left[\frac{\bar{d}^4}{\partial_-^2}\Delta_2, \frac{\bar{d}^4}{\partial_-}\Delta_4\right]_\star + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4) \Bigg\}. \tag{E.1}
\end{aligned}
$$

The last line in the equation above was dealt with in section 4.2.2. For the rest, the finiteness arguments are as follows. In line 1, we partially integrate out a factor d^4 (from the internal propagator) of either internal leg to the external legs. Note that these derivatives act in all possible ways on the two external legs producing odd phase factors due to equation (A.9). These phase factors would be potentially dangerous if we were combining terms to achieve finiteness. However, here the phase factors are irrelevant because each individual term is itself finite. In line 2, both internal legs carry a factor $\bar{d}^4$. Starting from either leg, this can be integrated out of the loop which becomes finite. The numerator in line 3 has a factor of p_- and a factor of q_- both of which do not contribute to the integral. Lines 4 and 5 both involve a factor of p_- in the numerator. These factors ensure that the integrals resulting from lines 3, 4 and 5 are finite.

As explained in section 4.2.2, the twenty-four contractions from the second quartic vertex in (4.18) reduce to twelve terms. Four of these twelve terms can be easily shown to

be finite using the manipulations described in the main text. This leaves eight terms

$$
\begin{aligned}
\mathrm{Tr}' \Bigg\{ &\left[\Delta_1, \frac{\bar{d}^4}{\partial_-^2}\Delta_2\right]_\star \left[\Delta_3, \frac{\bar{d}^4}{\partial_-^2}\Delta_4\right]_\star + \left[\Delta_1, \frac{\bar{d}^4}{\partial_-^2}\Delta_2\right]_\star \left[\Delta_4, \frac{\bar{d}^4}{\partial_-^2}\Delta_3\right]_\star \\
&+ \left[\Delta_2, \frac{\bar{d}^4}{\partial_-^2}\Delta_1\right]_\star \left[\Delta_3, \frac{\bar{d}^4}{\partial_-^2}\Delta_4\right]_\star + \left[\Delta_2, \frac{\bar{d}^4}{\partial_-^2}\Delta_1\right]_\star \left[\Delta_4, \frac{\bar{d}^4}{\partial_-^2}\Delta_3\right]_\star \\
&+ \left[\Delta_1, \frac{\bar{d}^4}{\partial_-^2}\Delta_3\right]_\star \left[\Delta_4, \frac{\bar{d}^4}{\partial_-^2}\Delta_2\right]_\star + \left[\Delta_1, \frac{\bar{d}^4}{\partial_-^2}\Delta_4\right]_\star \left[\Delta_3, \frac{\bar{d}^4}{\partial_-^2}\Delta_2\right]_\star \\
&+ \left[\Delta_3, \frac{\bar{d}^4}{\partial_-^2}\Delta_1\right]_\star \left[\Delta_2, \frac{\bar{d}^4}{\partial_-^2}\Delta_4\right]_\star + \left[\Delta_4, \frac{\bar{d}^4}{\partial_-^2}\Delta_1\right]_\star \left[\Delta_2, \frac{\bar{d}^4}{\partial_-^2}\Delta_3\right]_\star \Bigg\}.
\end{aligned}
\tag{E.2}
$$

The last two lines in the above equation were dealt with in section 4.2.2. Here, we briefly explain why the first two lines are finite. In the first term, we integrate the $\bar{d}^4$ away from Δ_4. If even one $\bar{d}$ is integrated to external leg 1, the term becomes finite. So we focus on the case where the four $\bar{d}$'s move to the other internal leg. This reads

$$
\left[\Delta_1, \frac{\bar{d}^4}{\partial_-^2}\Delta_2\right]_\star \left[\bar{d}^4\Delta_3, \frac{1}{\partial_-^2}\Delta_4\right]_\star,
\tag{E.3}
$$

In terms of momenta, the first line of (E.2) is now

$$
\left(\frac{1}{q_-^2}\frac{1}{l_-^2} - \frac{1}{q_-^2}\frac{1}{k_-^2}\right)[\Delta_1, \bar{d}^4\Delta_2]_\star [\bar{d}^4\Delta_3, \Delta_4]_\star.
\tag{E.4}
$$

Momentum conservation gives

$$
k = p + q - l, \qquad k = -l \ \text{for} \ l \gg p, q,
\tag{E.5}
$$

which implies that the divergent part of (E.4) vanishes. The proof of finiteness for the second line in (E.2) follows from similar arguments.

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